Thinking with shapes

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Is Mathematics Special?

- How is mathematics different from other enquiries?
- A mathematician is said to have answered, when asked whether he believed in God:

  “Yes, up to isomorphism.”
- So it’s something to do with structure, form or shape

We naturally turn to:

- Hilbert — Too many experts present!
- Bourbaki — Goes against our practice-oriented grain
- Resnik
- Shapiro

Both pre-occupied with ontology rather than practice

- Resnik, M. *Mathematics as a Science of Patterns* (Oxford: Clarendon, 1997)
- Shapiro, S. *Philosophy of Mathematics: Structure and Ontology* (New York: OUP, 1997)

Both directed at established mathematical knowledge rather than mathematics in the making.

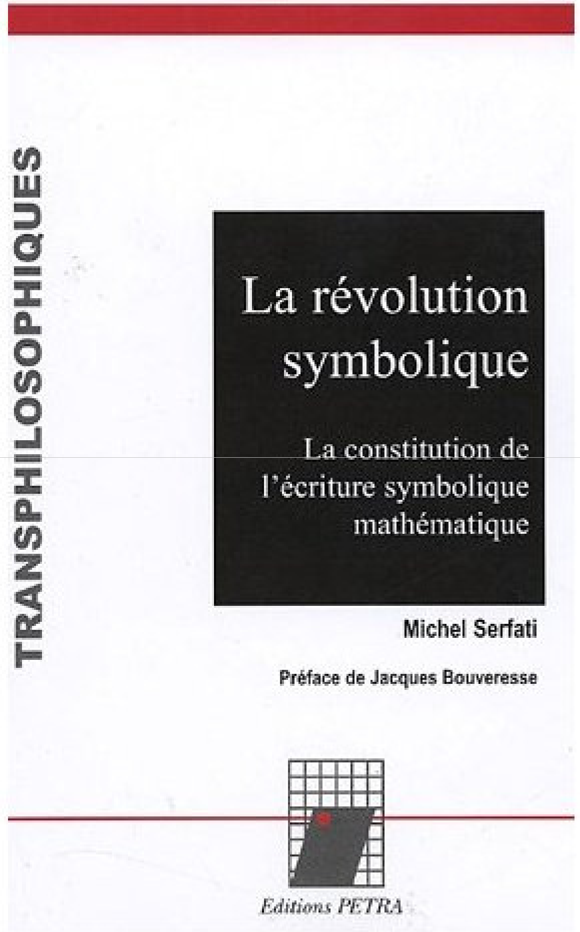
Claim

Pattern, form, structure and (generally) shape have roles in mathematical practice (and, arguably, in mathematical practice *only*) that escape the attention of ontologically-focused structuralisms

(But may contribute to the plausibility of these views)

Speculative Corollary

If these heuristic uses of structure really are unique to mathematics, they may offer a clue to its unique epistemological status.



Drawing on Serfati:

Shapes in notation

Two sorts of case

- Substitution
- Extension

There are lots of others…

A case of substitution:

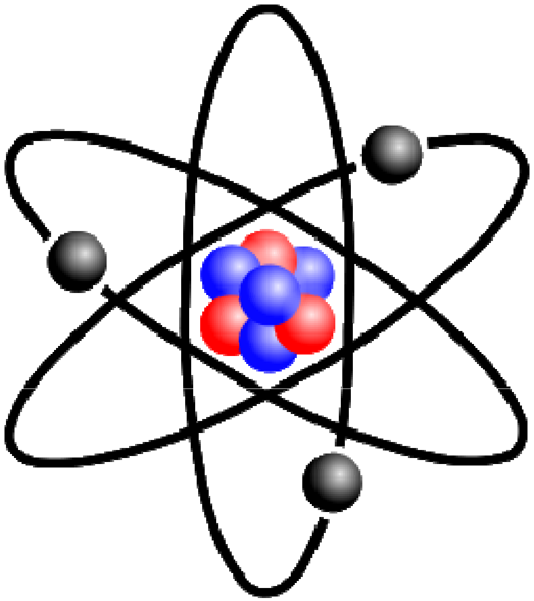
Leibniz's general product formula

$$(x+y)^n = x^n y^0 + nx^{n-1} y + \frac{n(n-1)}{1.2} x^{n-2} y^2 + \ldots + x^0 y^n$$

$$p^n(x+y) = p^n x p^0 y + np^{n-1} x p^1 y + \frac{n(n-1)}{1.2} p^{n-2} x p^2 y + \ldots + p^0 x p^n y$$

$$d^n(xy) = d^n x d^0 y + nd^{n-1} x d^1 y + \frac{n(n-1)}{1.2} d^{n-2} x d^2 y + \ldots + d^0 x d^n y$$

There are structural analogies in natural science



Rutherford ‘planetary’ model of lithium

…but in the notation ?

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Seven Extensions

1. Factorial of a positive real number
2. Exponential of a complex number or square matrix
3. Trigonometric functions of complex numbers
4. Matrix pseudo-inverses
5. Derivative of a non-differentiable function
6. Derivative of a function on normed vector-spaces
7. Union and intersection of r-partitions

Case 1: Factorial of a positive real number

- How can n! mean anything unless n is a natural number?
- Ask Euler!
- n!= n.(n-1)!

$$\Gamma(x) = x.\Gamma(x-1)$$

$$\Gamma(x) = \frac{1}{2}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{x}\left(1+\frac{x}{n}\right)^{-1}\right\}$$

$$\Gamma(x) = \lim_{n\to\infty}\left(\frac{1.2\ldots(n-1)}{x.(x+1)\ldots(x+n-1)}n^{x}\right)$$

…which satisfies the functional equation and equates to x! when x is a natural number

Case 2: Exponential of a complex number or square matrix (Euler again)

$$e^{a+b.\sqrt{-1}}$$ *a*, *b* both real

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$e^{a+b.\sqrt{-1}} = 1 + \frac{a + b.\sqrt{-1}}{1!} + \frac{\left(a + b.\sqrt{-1}\right)^2}{2!} + \ldots + \frac{\left(a + b.\sqrt{-1}\right)^n}{n!} + \ldots$$

…then treat complex numbers as square matrices

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Case 3: Trigonometric functions of complex numbers (still Euler)

$$\cos\left(a + b.\sqrt{-1}\right)$$ *a*, *b* both real

$$\cos(x) = \sum_{n \in \mathrm{N}} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos\left(a + b.\sqrt{-1}\right) = \sum_{n \in \mathrm{N}} \frac{(-1)^n \left(a + b.\sqrt{-1}\right)^{2n}}{(2n)!}$$

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Case 4: Matrix pseudo-inverses

- Can we give a sense to A^{-1} when A is not invertible?
- Yes, and there are several options (unlike in physics…)
- All coincide with the usual inverse in the case of invertible matrices

Moore-Penrose pseudo inverse

Four trivial identities for invertible matrices:

$$AA^{-1}A = A$$

$$A^{-1}AA^{-1} = A^{-1}$$

$$(AA^{-1})^* = AA^{-1}$$

$$(A^{-1}A)^* = A^{-1}A$$

Become the definition of the pseudo-inverse

(* is the adjoint)

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Moore-Penrose pseudo inverse

Penrose proved that for every matrix A there is a unique matrix X such that:

$$AXA = A$$

$$XAX = X$$

$$(AX)^* = AX$$

$$(XA)^* = XA$$

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Another pseudo-inverse

Provide the vector space of matrices with a norm. Note that if a square matrix is invertible, its inverse is the unique matrix B such that:

$$\|AB - I\| = 0$$

Again, we take a trivial identity in the original case, and make it the definition of the extension.

Another pseudo-inverse

Now let A be any complex matrix. The right-hand pseudo-inverse is the unique matrix B such that:

$$\left\| AB - I \right\| \quad \text{is minimal}$$

Note how this reduces to the original case (=0), and proving uniqueness is again the hard part

Case 5: Derivative of a non-differentiable function

- Integration by parts plays the role of the ‘bridge’
- Thus differentiation is extended to locally integrable functions
- Laurent Schwartz won a Fields medal for it…

Case 6: Derivative of a function on normed vector-spaces

The derivative is originally conceived at the limit of a quotient:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

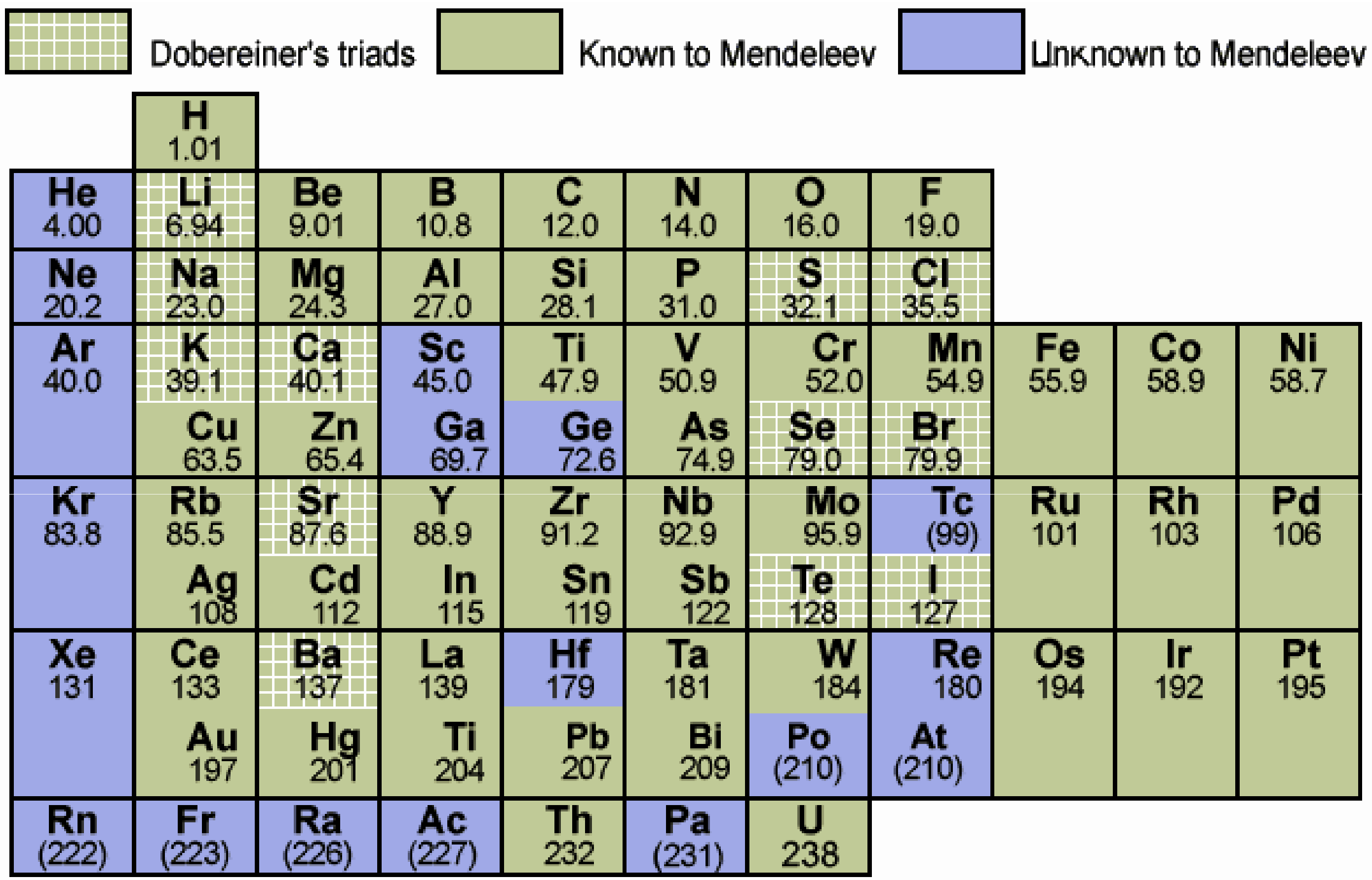
But vectors don't form quotients… so:

$$f(a+h) = f(a) + f'(a)h + o(\|h\|)$$

Case 7: Union and intersection of r-partitions

Enough already!

There are cases of pattern-completion in natural science:



But without the mathematician’s latitude

(recall the choice of pseudo-inverses)

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