Contradictions in Mathematics

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Topics

- Byers on Contradictions in Mathematics
- Inconsistent Mathematics
- The Benefits of Inconsistent Arithmetic

Introduction

- Contradictions are typically seen as anathema to mathematics. As formalism sees consistency as the only condition to consider some mathematical structure as an object of study, inconsistency becomes the single exclusory criterion.
- For mathematical realists accepting inconsistencies comes down to accepting inconsistent objects, which just seems bizarre.
- In this paper I consider two inconsistency friendly approaches in (the philosophy of) mathematics. In a recent study *How Mathematicians Think* William Byers argues that one way of mathematical progress is by way of contradiction. The first paragraph outlines Byers' thesis, but it turns out that contradictions play a role only *ex negative*.
- In contrast to that the approach of inconsistent mathematics claims contradictions to be real. Especially in inconsistent arithmetic contradictions are said to play a vital role. They turn out to provide a framework for a finitist position which endorses inconsistent numbers.

Byers‘ Approach

- William Byers claims mathematics to at core a creative activity. Mathematical reasoning, according to Byers, is not primarily algorithmic or based on proof systems, but on using (great) ‘ideas’ to shed new light on mathematical objects and structures. These ideas not only are placed at the centre of mathematical understanding, which Bryer calls ‘turning on the light’, but also propel mathematical progress.
- Bryers presents a couple of examples in which a crucial step forward in the development of mathematics depended on the presence of two at first sight unrelated or even barely compatible perspectives on some mathematical structure. He starts with the discovery of the irrational numbers (like $\sqrt{2}$), where $\sqrt{2}$ is clearly present as a geometric object (the length of the hypothenuse of the right angled triangle with unit length sides) but is not allowed for by (early Greek) arithmetic. The real numbers ‘provide a context’ (38) in which the two perspectives are unified. [Another famous example is the *Fundamental Theorem* of the calculus, which says ‘that there is in fact one process in calculus that is integration when it is looked at in one way and differentiation when it is looked at in another’ (50).]

Byers‘ Approach (II)

- The core of mathematics, according to Bryers, is finding such situations and being able to understand them by providing a more comprehensive view. This process is creative and not algorithmic. Proofs only sum up the discovery and preserve the result in text books. Mechanical proofs Bryers sees as ‘trivial’ (373) whereas ‘deep’ proofs are framed in expressing some (great) ‘idea’ (like re-ordering infinite series makes it obvious to see a sum formula).
- Good mathematicians are, therefore, those who hit on ‘ideas’ (like Cantor hitting on diagonalization and the continuum hypotheses).
- Even more revolutionary are ‘great ideas’. An example of a great idea is formalism. Formalism provided a unifying perspective on the whole of mathematics.

  [When Hilbert started with formalizing Euclid’s geometry ‘*formalism* was born and, in the process, the whole notion of truth was radically transformed’ (291). A great idea is then inflated (like in Hilbert’s claims on behalf of formalism) and then again delimited in a wider perspective (like when Gödel’s Theorems hit formalism).]

Byers on Contradiction

- One of the central methodological concepts – besides ambiguity – Byers uses is ‘contradiction’.
  The very subtitle of his book reads ‘Using Ambiguity, Contradiction, and Paradox to Create Mathematics’.

- ‘Contradiction’ is understood by Byers in two ways.

  1. On the one hand we have two seemingly contradictory perspectives in some of the mathematical problems he presents.

  For example, one may see $\sqrt{2}$ as a decimal, ‘an “infinite” indefinite object’ (97), but also as a finite geometric object. One can see ‘2 + 3 = 5’ both as expressing a fact of identity (i.e. something static) as well as expressing the process of adding (i.e. something dynamic).
  But of course the fact can be established by going through the process of adding, the two perspectives are finally compatible and not inconsistent.
  The paradox of zero (as something that is nothing) vanishes with axiomatization.

Byers on Contradiction (II)

2. On the other hand some seminal proofs work by using contradictions, or so Byers claims.

For a simple example, one can argue for the proposition that a straight lien falling on parallel straight lines makes the alternate angles equal to one another in the following fashion: ‘Suppose on the contrary, that angle α is not equal to angle β, for example, angle α is smaller than angle β. Then by adding angle γ we end up with

$$\text{angle } \alpha + \text{angle } \gamma < \text{angle } \beta + \text{angle } \gamma = \text{two right angles.}$$

Thus the interior angles on the same side are less than two right angles. The parallel postulate tells us that in this situation the lines must meet, contradicting the assumption that they are parallel.’ (95).
Other famous examples are Cantor’s use of diagonalization or Gödel’s Theorems.

Byers‘ Contradictions Aren‘t Real

- Now, if we look at these examples it becomes obvious that none of the mathematicians in question endorses any of the contradictions. Quite the opposite. In these indirect proofs contradictions are used as a threat to establish the opposite result. Just for reduction some innocent looking assumption is made which turns out to be contradictory and thus untenable. What we really see here is not a creative use of contradictions, but the creative use of indirect proof methods. Mathematics still avoids the contradictory.
- Even supposedly incompatible perspectives on one and the same structure have to be kept distinct from contradictions. The paradoxical calls for resolution. The perceived incompatibility is the very reason to look for another solution.
- At last Byers admits ‘we cannot leave it at that – things *must* be reconciled’ (111). ‘New stages of mathematical development arise out of a “resolution” of a set of paradoxes’ (186).

Inconsistent Mathematics

- To have *an inconsistent number theory* means at least that within the theorems of number theory there is some sentence A with A being a theorem and $\neg$A being a theorem at the same time. Supposedly this contradiction corresponds to some object/number *o* being an inconsistent object. So inconsistent mathematics is connected to inconsistent ontology.
- Its underlying logic has to be paraconsistent (cf. Bremer 2005). Changing the basic logic makes mathematics in a weak sense paraconsistent: If there were to turn up some inconsistency in mathematics, it would not explode.

Inconsistent Mathematics (II)

- The problems with having F(a) and ¬F(a) for some object *a* seem not so pressing if *a* is some mathematical object than *a* being a physical object Mathematical objects are either non-existent – *mere* theory, taken instrumentally – or they are in some elusive Platonic realm where strange things may well happen.
- If on the other hand one is a reductionist realist about mathematics (mathematics being about structures of reality or mathematical entities rather being concrete entities dealt with by mereology) then inconsistent mathematics is as problematic as your cat being (wholly) black and not being (wholly) black at the same time.

Inconsistent Mathematics (III)

- Philosophers when concerned with mathematics focus on number theory, since the ontological questions of mathematics ("What and where are mathematical objects?", "Are there infinite sets?"...) and the epistemological questions of mathematics ("How do we know of numbers?", "Is mathematics merely conventional?"...) do arise already with number theory.
- Taking set and model theory as part of logic anyway, logicians are also mainly concerned with number, since a lot of meta-logical theorems make us of the device of arithmetization. The same goes for the general theory of automata and computability. I follow this focus here and so this paragraph concerns itself mostly with arithmetic. This may not be enough for a mathematician trying to assess the power of inconsistent mathematics. She looks for inconsistent theories at least of the power of the calculus. There are actually such theories, e.g. presented by Chris Mortensen (1995).

Using *N*

- One of the most fundamental mathematical theories is arithmetic (as given for instance by the Peano axioms). We are concerned mainly with first order representations. In distinction to an axiomatic arithmetic theory like Peano Arithmetic there is the arithmetic *N* (being the set of true first order arithmetic sentences in the standard interpretation).
- *N* is negation complete (either A or ¬A is in *N*), not axiomatisable, not decidable, and, of course, infinitely large.
- Given the first order representation or arithmetic there are a lot of well-known theorems about arithmetic, e.g. (first order) arithmetic being incomplete.

Using Less Numbers

- Given Compactness of **FOL** one can prove that there are non-standard models of arithmetic, which contain additional numbers over and above the natural numbers. These additional numbers behave consistently, however. Consistency provides them in the first place.
- Inconsistent arithmetic may concern itself with the opposite deviance: Having arithmetics where there are less numbers than in standard arithmetic.
- This is of outmost philosophical interest, since the infinite is a really problematic concept leading to the ever larger cardinalities of "Cantor's paradise", and finitism (in the sense of the assumption that there are only *finitely many* objects, even of mathematics) is therefore an option worth exploring and pursuing.

Inconsistent Arithmetic

- Robert Meyer was the first to give a non-triviality proof of a Relevant (paraconsistent) arithmetic. The system **R#** is an extension of the first order version of Relevant logic **R** with axioms mirroring those of Peano arithmetic save that the "$\supset$" in them has been replaced by the Relevant "$\rightarrow$". Induction is present as a rule. **R#** is non-trivial in that 0=1 is not provable. This non-triviality can be established by finitistic methods.
- Inconsistent Arithmetic that are finite may have any finite size you like. They contain one largest number. Since we do not know which number really is the largest we may assume that one of these arithmetics is true, although we don't know which. Which one it is is not that important, since all these arithmetics have common properties:

Properties of Finite N-Variants

Let *n* be some natural number.

Let *N*n be a set of arithmetic sentences.

n is the supposed largest number of *N*n.

These sets/theories have the following properties (cf. Priest 1994):

(i) $N \subset N\mathrm{n}$.

(ii) *N*n is inconsistent.

(iii) $A \in N\mathrm{n}$ for a (negated) equation A concerning numbers $< n$ if and only if $A \in N$.

(iv) *N*n is *decidable*.

(v) *N*n is *representable* in *N*n (thus we have a *N*n truth predicate).

(vi) For the proof predicate B( ) of *N*n every instance of B("A") ⊃ A is in *N*n. (Löb-Property)

(vii) If A is not a theorem of *N*n ¬B("A") ∈ *N*n.

(viii) For the Gödel sentence G for *N*n $G \in N\mathrm{n}$ and $\neg G \in N\mathrm{n}$.

Properties of Finite N-Variants (II)

An inconsistent arithmetic *N*n thus has quite remarkable properties:

- by (i) we have that *N*n is complete, since *N* is.
- by (ii) and (viii) we have, of course, that it is inconsistent.
- by (iv) it has all the nice properties that *N* does not have, although *N*n is complete!
- by (v) we can in the language of arithmetic define a truth predicate for that very same language.
- by (vi) *N*n has an ordinary proof predicate.
- by (vii) in conjunction with (iii) we have not only that *N*n is not trivial (by excluding some the equations that are excluded by *N*), but that this non-triviality can be established within *N*n itself.

Proof Outline for *N*n-Properties

- A theory with less numbers than *N* can have less counterexamples to a given arithmetic sentence. Thus it contains at most more sentences (as true). This holds in general (called "Collapsing Lemma"). Therefore (i). So we do not lose any of the power of *N* by switching to *N*n.
- Since *N* is negation complete *adding* any sentence (as true) means adding a sentence for which the negation is already in *N*. Thus the resulting theory contains for at least one A, A and ¬A. Thus (ii). This means that the logic of these arithmetic theories has to be a paraconsistent logic.
- Representability of truth is a consequence of (iv) and (i). The same holds for the representability of the proof predicate, (vi). Once the proof predicate is representable in the decidable theory *N*n we can represent non-provability, and thus have (vii) and finally (viii).
- (iii) is the most interesting property and results from the way the domain of a corresponding model is constructed.

Finite Models

- A model of a theory *N*n is constructed as a filtering of an ordinary arithmetic model. In general one can reduce the cardinality of some domain by substituting for the objects equivalence classes given some equivalence relation (i.e. instead of objects o1, o2 ... we have [o1], [o2]...). The equivalence classes provide then the substitute objects. Since the objects within the equivalence class are equivalent in the sense of interest in the given context the predicates still apply (now to the substitute object).
- The trick in case of *N*n is to chose the filtering which puts every number $< n$ into its equivalence class, and nothing else; and puts all numbers $\geq n$ into n's equivalence class. As a result of this for $x < n$ the standard equations are true (of [x]), while in case of $y \geq n$ *everything* that could be said of such a y is true of [n].
- So we have immediately $n = n$ (by identity) and $n = n + 1$ (since for $y = n + 1$ in *N* this is true).

Finite Models (II)

- The domain of a theory *N*n so is of cardinality n.
- n now is an inconsistent object of *N*n.
- If for the moment we picture the successor function by arrows we can picture the structure of a model of *N*n thus:

↶

$$0 \to 1 \to ... \to n$$

- Such models are called "heap models".
- The logic modelling *N*n has to be paraconsistent. And it has to have restrictions on standard first oder reasoning as well.

RM3 Arithmetic

- Mortensen chooses **RM3#** as basic system and finitizes it by substituting for a number n the number n *modulo* some m. Thus the domain becomes {0, 1, 2, ... m-1}. The resulting arithmetic **RM3m** is complete, non-trivial and decidable. **RM3m** is *axiomatisable* by adding to **RM3#** the axioms:

$$\vdash 0 = m$$

- and all instances of the following axiom scheme for $n \in \{0, 1, ... m\text{-}1\}$:

$$\vdash (0 = n \leftrightarrow 0 = 1).$$

- The approach "*modulo* some m" has at least the same deviant results than the heap models mentioned before: In **RM35** we have $4 + 2 = 6$ (since **RM35** is complete, i.e. has all theorems of *N*) and $4 \times 6 = 4$ (since "6" denotes 1). And the approach "*modulo* some m" has these deviant sentences for some *known* numbers!

Finitization

- Arithmetic is constructed thus as a finite theory. One can generalized the steps of this procedure to apply it to other mathematical theories. Van Bendegem distinguishes the following steps:

  (i) Take any first-order theory T with finitely many predicates. Let M be a model of T.

  (ii) Reformulate the semantics of T in a paraconsistenf fashion (i.e. the mapping to truth values and overlapping extensions of P+ and P-).

  (iii) If the models of M are infinite, define an equivalence relation R over the domain D of M such that D/R is finite.

  (iv) The model M/R is a finite paraconsistent model of the given first-order theory T such that validity is at least preserved.
- The restriction to theories with finitely many predicates is no real restriction in any field of applied mathematics or formal linguistics, since no physical device (be it human or machine) can store a non-enumerable list of basic predicates.

Paraconsistent Löwenheim/Skolem

- The *Löwenheim/Skolem-Theorem* is one of the limitative or negative meta-theorems of standard arithmetic and **FOL**. It says that any theory presented in **FOL** has a *denumerable* model. This is strange, since there are first order representations not only of real number theory (the real numbers being presented there as uncountable), but of set theory itself. Thus the denumerable models are deviant models (usually Herbrand models of self-representation), but they cannot be excluded.
- Given the general procedure to finitize an existing mathematical first order theory using paraconsistent semantics, there is a paraconsistent strengthened version of the *Löwenheim/Skolem-Theorem*:

  Any mathematical theory presented in first order logic has a *finite* paraconsistent model.

The Benefits of Inconsistency

- A mathematics that does not commit us to the infinite is a nice thing for anyone with reductionist and/or realist leanings. As far as we know the universe is finite, and if space-time is (quantum) discrete there isn't even an infinity of space-time points. The largest number may be indefinitely large. So we never get to it (e.g. given our limited resources to produce numerals by writing strokes).
- If there is a largest number n there is the corresponding inconsistent arithmetic *N*n. We can presuppose *N*n being our arithmetic. Since *N* and *N*n agree on all finite and computational mathematics it is hard to see whether we lose anything important at all by switching to *N*n. If we have paraconsistency anyway for other reasons, we get this finitism *for free*, it seems. So why not take it? In as much as *N*n is correct no correct reasoning transcends the finite.
  [Hilbert wouldn't have rejoiced, probably, since *N*n of course is inconsistent itself. The drawback of all this is, of course, the problem of an ontology of inconsistent entities – at least if you are a realist.]

The Benefits of Inconsistency (II)

- If there are inconsistent versions of more elaborated mathematical fields like the calculus one may draw some general philosophical conclusions:

  - If there are corresponding inconsistent versions of these mathematical theories with comparable strength to the original theories then consistency is not the fundamental mathematical concept, but functionality (of the respective basic concepts) may well be.

  - If the justification of mathematics depends on its applicability and the inconsistent versions are of comparable applicability then they are justified not just as mathematical theories, but even in the wider perspective of grasping fundamental structures of reality; there no longer will be available the argument from mathematical describability to the consistency of the world.

References

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